Theoretical prediction of liftoff angular velocity distributions of sand particles in windblown sand flux

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The liftoff angular velocities, including the rebound ones and the ejected ones, of sand particles in a windblown sand flux have a significant influence on the saltation trajectories and the transport flux and so on. However, they are not easy to determine actually both in field and in a wind tunnel, especially when the sand particles are small and the wind speed is low. A stochastic particle-bed collision model is developed in this paper to calculate the liftoff angular velocities. Having taken the incident velocity and angle of an incident particle, the impact angle between the incident and the impacted particles in the sand bed and its creep velocity as well as the contact angle between the impacted particle and other particles in the bed as random variables in the collision model, the probability distributions of the liftoff angular velocities for different particle diameters and wind speeds are gained and agree well to the experimental results. The rebound probability distribution function looks like Gumbel distribution with a cusp in the peak, and the ejected one looks like the normal one. The effect of the wind speed and the particle diameter on the probability density functions is discussed in detail.

as well as the contact angle between the impacted particle and other particles in the bed, as random ones, the probability density functions of the liftoff angular velocities are derived in section 3. The measurement data of the angular velocities of saltating particles in a windblown sand flux are given in section 4 to partly check the availability of the theoretical prediction. Some results and discussions are presented in section 5 and conclusions are listed in section 6.

2. Collision Model and Formulas of the Angular Velocities

[6] The collision between an incident sand particle and an impacted one in a sand bed, denoted respectively by the particles A and B, is simplified as an interaction among three particles and shown in Figure 1. The supporting forces arisen from all other particles touching with the particle B in the sand bed are incorporated into a resultant one sustained by a particle C at the sustaining point J2. The particles are regarded as 2D disks usually adopted in many previous researches [Ungar and Haff, 1987; Werner and Haff, 1988; Anderson et al., 1991; Anderson and Haff, 1988]. Before the collision, the particle B with radius $R_B$ creeps in the bed with a creep velocity $V_B$ and will be impacted by the particle A with radius $R_A$ at the point $J_1$. The incident velocity and angle are respectively denoted by $V_A$ and $\theta_0$. The angle of the line joining the particles A and B’s centers to the horizontal (x axis) and the angle of the line joining the particles B and C’s centers to the vertical (y axis) are respectively called the impact angle $\beta$ and the contact angle $\alpha$ shown in Figure 1. It is obvious that not only the positions of the points $J_1$ and $J_2$ but also the incident velocity $V_A$ and the incident angle $\theta_0$ as well as the creep velocity $V_B$ are quantitatively undetermined in the collision. So, the angles $\alpha$, $\beta$, $\theta_0$ and velocities $V_A$ and $V_B$ are regarded as random variables in the particle-bed collision model. Their ranges and probability density functions can be known with the aid of some experiment results and geometry analysis. Moreover, the incident velocity $V_A$ and the contact angle $\alpha$ as well as the creep velocity $V_B$ are independent except that the impact angle $\beta$ is related to the incident angle $\theta_0$ according to Rumpel’s [1985] analysis.

[7] Having assumed that the collision among the sand particles is non-elastic and the deformations of all contact points of the particles reach their maximum values at the same time, 12 algebraic equations (see Appendix A) can be established respectively for the compress and the recovery process of the collision between the particles A and B based on the particle-bed collision model suggested in this paper. Having solved these 12 equations, the analytic expressions of the liftoff velocities of A and B can be obtained, here, which are denoted respectively by the rebound angular velocity $\omega_A$ and the ejected angular velocity $\omega_B$, are listed as follows

$$\omega_A = \left\{ \begin{array}{l} 2R_A(1 + \sin(\alpha + \beta))[\cos(\alpha + \theta_0) - \sin(\beta - \theta_0)] \\ - 3\sin(\beta - \theta_0)2(1 + k)\sqrt{\frac{9R_AR_B^3 + 4R_A^3[1 + \sin(\alpha + \beta)]}{9R_A + 4R_A^3[1 + \sin(\alpha + \beta)][2 - \sin(\alpha + \beta)]/R_B^3}} \end{array} \right. \quad (1)$$

Figure 1. Schematic of the particle-bed collision model.

Figure 2. Probability densities of incident velocity and incident angle under wind speed 8.0 m/s and particle diameter 0.25 mm. (a) Incident velocity and (b) incident angle.
where $\omega_B = -[\sin(\beta - \theta_0) + \cos(\alpha + \theta_0)] 
\cdot [2(1 + k)V_A/\{9R_B + 4R_A^2[1 + \sin(\alpha + \beta)] 
\cdot [2 - \sin(\alpha + \beta)]/R_B^2\} - (1 + k)V_B/(9R_B) 
+ 4R_A^2[1 + \sin(\alpha + \beta)] - 2 - \sin(\alpha + \beta)]/R_B^2\} \right) 
\right)

in which $k$ is a restitution coefficient and taken as a constant for the quartz sand. From (1) and (2), it can be known that the liftoff angular velocities, $\omega_A$ and $\omega_B$, are the functions of random variables $\alpha$, $\beta$, $V_A$, $V_B$, and $\theta_0$ after given particle radius $R_A$ and $R_B$ as well as the restitution coefficient $k$. In addition, the liftoff angular velocities are also related to the wind speeds because the incident velocity $V_A$ and the incident angle $\theta_0$ as well as the creeping velocity $V_B$ vary with the wind speed.

\begin{equation}
f_{ca} = \int \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B d\theta_0 \tag{3}
\end{equation}

\begin{equation}
f_{ca} = \int \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B d\theta_0 \tag{4}
\end{equation}

where $|J_{ca}| = |\partial \omega_A/\partial V_A|$ and $|J_{ca}| = |\partial \omega_B/\partial V_B|$ are respectively Jacobian determinant, $f_{\omega_A} V_A \omega_B \omega_A$ can be expressed by the probability density functions $f_{ca}$, $f_{\omega_A}$ and $f_{\omega_B}$ of the variables $\alpha$, $\omega_A$, and $\omega_B$ and the joint probability density function $f_{\omega_0}$ that is $f_{\omega_0} V_A \omega_B \omega_A = f_{\omega_A} V_A \omega_B \omega_A \cdot f_{\omega_B} V_A \omega_B \omega_A$ because of their independence. Due to $f_{\omega_0} = f(\beta | \theta_0) f(\omega_A | \theta_0) f(\omega_B | \theta_0)$ in which $f(\beta | \theta_0)$ is the conditional probability density function of the impact angle $\beta$ for a given incident angle $\theta_0$ and $f(\omega_A | \theta_0)$ is the probability density function of $\theta_0$, the probability density functions of the rebound and ejected liftoff angular velocities can be solved by integrating formulas (3) and (4) once $f_{\omega_A}$, $f_{\omega_B}$, $f_{\omega_A}$, and $f_{\omega_B}$ are determined.

[9] The probability distribution functions of the contact and the impact angles $\alpha$ and $\beta$ are firstly discussed. It is obvious that the range of the contact angle $\alpha$ is related to the arrangement pattern of the sand particles in a sand bed. The widest range for $\alpha$ is $(-\pi/2, \pi/2)$ when the arrangement pattern is a packed one [Anderson and Haff, 1988]. The range of the impact angle $\beta$ is, as mentioned above, related to the incident angle $\theta_0$. When the particle $A$ impacts the sand particle $B$ downwind ($\theta_0 \leq \pi/2$), the impact angle $\beta$ is usually located in the range $[\theta_0, \theta_0 + \pi/2$] [Rumpel, 1985; McEwan et al., 1992]. Similarly, the impact angle $\beta$ is usually located in the range $[\theta_0 - \pi/2, \theta_0]$ when the collision happens in the lee side of the particle $B$ ($\theta_0 \geq \pi/2$). Having assumed the angle $\alpha$ and the angle $\beta$ uniformly distribute in their ranges, their probability distribution functions can be respectively expressed as follows

\begin{equation}
f_{\alpha} = 1/\pi, \; f_{\beta}(\beta | \theta_0) = 2/\pi \tag{5}
\end{equation}

Having substituted the expression (5) and the expressions of $|J_{ca}| = |\partial \omega_A/\partial V_A|$, $|J_{ca}| = |\partial \omega_B/\partial V_B|$ into the expressions (3) and (4), the probability density functions of the liftoff angular velocities $\omega_A$ and $\omega_B$ are rewritten as follows

\begin{equation}
f_{ca} = \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B \tag{6}
\end{equation}

\begin{equation}
f_{ca} = \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B \tag{7}
\end{equation}

3. Theoretical Modeling and Predictions

[8] With the aid of the probability theory of random variable [Bickel and Doksum, 1977], the probability density functions of the liftoff angular velocities $\omega_A$ and $\omega_B$ can be obtained by following expressions, that is

\begin{equation}
f_{ca} = \int \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B d\theta_0 \tag{3}
\end{equation}

\begin{equation}
f_{ca} = \int \int \int \int f_{ca} V_A \omega_B \omega_A dV_A d\omega_A d\omega_B d\theta_0 \tag{4}
\end{equation}

\begin{equation}
f_{\omega_A} = \begin{cases}
-1/4 V_B + 1/2 & 0 \leq V_B \leq 2.0 \\
V_B^2 + 1/2 & -2.0 \leq V_B \leq 0
\end{cases} \tag{8}
\end{equation}

Since it is not easy to analytically express the probability density of the incident velocity $V_A$ and the incident angle $\theta_0$, their numerical ones for a given wind speed and a given particle diameter 0.25 mm are respectively plotted in Figures 2a and 2b based on the experimental data measured by Phase Doppler Anemometry (PDA) 1 mm close to a sand bed in a wind tunnel [Dong et al., 2004].

[11] It can be found, according to above discussions, that a numerical integration is necessary for the expressions (6) and (7) to obtain the probability densities of the liftoff angular velocities $\omega_A$ and $\omega_B$. It should be noted that $V_B$ in the expressions (6) and (7) can be respectively replaced by $\omega_A$ and $\omega_B$ with the aid of the expressions (1) and (2). Since the probability density functions of the incident velocity $V_A$ and the incident angle $\theta_0$ vary with the wind speed and the particle diameter, the probability density functions $f_{ca}$ and $f_{\omega_A}$ are related to the wind speed and particle diameter. The probability density function $f(\omega)$
of the liftoff angular velocity \( \omega \) of the particles can be obtained as follows

\[
f(\omega) = \alpha f_{wA} + \beta f_{wB}
\]  

(9)

in which \( \alpha \) is the rebound probability for the particle \( A \). According to Mitha's experimental conclusion that the rebound probability in a steady windblown sand flux is about 0.94, we have \( \alpha = 0.94 \) and \( \beta = 0.06 \) due to \( \alpha + \beta = 1 \).

4. Experimental Confirmations

[12] The validity of the theoretical prediction of the liftoff angular velocities and the formulas (6)–(7), (9) are firstly respectively checked by Tanaka and Kakinum's experimental values and Chepil's ones. Since they just gave some more possible ranges of the angular velocity, the probabilities of the liftoff angular velocity predicted in these ranges are respectively calculated by the formulas (6)–(7), (9). From Table 1, it can be found that the probabilities of the liftoff angular velocities located in the ranges 245–780 rev/s and 280–820 rev/s measured by Tanaka and Kakinum are respectively 0.5789 and 0.5237, and the probability of angular velocities in the range 200–1000 rev/s measured by Chepil is 0.631. Compared with the probabilities of liftoff angular velocities in other ranges out of experimental ones, the probabilities in the experimental ranges are dominant. It means that the main ranges of the predicted liftoff angular velocity are in accord with the experimental ones.

4.1. Test Apparatus and Methodology

[13] Since the pervious experimental values of the angular velocity, mentioned above, are not enough to give a detail confirmation for the predicted liftoff angular velocities and their probability densities, an experiment was carried out in this paper to measure the angular velocities of saltating sand particles in a windblown sand flux in a wind tunnel at room temperature 22\(^\circ\)C at night (pithy-dark environment).

[14] The mean diameter of sieved natural sand particles is 0.25 mm and the shear speed is respectively kept on 0.67 m/s, 0.83 m/s and 0.87 m/s in the experiment. The schematic diagram of the experimental apparatus is shown as Figure 3. After the wind speed was steady, MS-230 light source, as the high-speed multi-flash light source, irradiated the sand bed vertically from the top of the wind tunnel and formed a 1-cm wide light spot, and a FUJICA-135 camera photographed the saltating trajectories of sand particles out side of the wind tunnel and its exposure times is 0.002 s. And many images of trajectories of sand particles spinning with any angular velocities can be obtained.

4.2. Test Results and Errors Analysis

[15] With the exposure time adopted in the experiment, the motion of the sand particle formed a short, bar-like image on the film (Figure 4). Since natural sand particles are not spherical, the bar-like image exhibits a spiral shape during rotation and migration [Chepil, 1963; Éwannouve, 1972]. Because a spiral image represents a complete cycle, when a bar-like image occurs \( \Omega t \) cycles on the film, the sand particle is actually rotating at \( \Omega t \) rev/s where \( t \) is the exposure time of the camera, so the angular velocity of the sand particle is 5000 \( \omega \) rev/s. And at shear speed of 0.67 m/s,
Here, we randomly selected the images of 20 sand particles from three sets of photographs, for which the shear speeds are respectively taken as 0.67 m/s, 0.83 m/s and 0.87 m/s, and viewed them under magnifying glass at a magnification of 5 times to estimate the errors of the experiment. The result showed a largest relative error for Figure 5.

Comparison between theoretical and experimental probabilities of angular velocities, where the heights and shear speeds are respectively (a) 0–1 cm, 0.67 m/s, (b) 1–2 cm, 0.67 m/s, (c) 0–1 cm, 0.87 m/s, (d) 1–2 cm, 0.87 m/s, (e) 2–3 cm, 0.87 m/s and (f) 3–4 cm, 0.87 m/s.
measured angular velocities is less than 15%. Thus, the measured results are credible.

[17] The experimental angular velocities are divided into categories with 1 cm height intervals, and the probability profiles of the angular velocities of the saltating sand particles in each height cell are obtained by statistic. The statistical results are listed in Table 1.

[18] Since the angular velocities measured in the experiment are the ones of particles located 1–5 cm in height above the sand bed, the calculated values of the liftoff angular velocities have to be extrapolated to the ones at heights corresponding to the experiment. With the aid of a saltation model [White and Schluž, 1977], in which the probability density functions of the liftoff velocity and the liftoff angular velocity, as initial conditions of the model, are respectively taken one given by Anderson and Hallet [1986] and one presented by the formulas (6)–(7) and (9) in this paper, the theoretical values of the angular velocities in a given height cell can be obtained and their probability profiles under the experimental condition are plotted in Figure 5. Although there exists some derivation in quantity, it can be found, from Figure 5, that the probability profiles of the predicted angular velocities have a good agreement with the experimental ones in the range, shape, skewness and peak value. Moreover, it can be found that the derivation becomes small with the increasing wind speed because the trajectories are easier to distinguish in high wind speed than in low one. So, it can be inferred that the derivation

Figure 6. Probability densities of rebound and ejected angular velocities with wind speed.

Figure 7. Probability densities of angular velocities under different particle diameters. (a) Ejected angular velocity and (b) rebound angular velocity.

Figure 8. Probability densities of angular velocities under different wind speeds. (a) Ejected angular velocity and (b) rebound angular velocity.
mainly arises from the limitation of the measurement instrument in experiment.

5. Results and Discussions

[19] Having confirmed the validity of the predicted liftoff angular velocities, the probability densities of both the rebound and ejected liftoff angular velocities are calculated respectively by the formulas (6) and (7). Figure 6 displays the probability densities of the rebound and ejected liftoff angular velocities for the wind speed 8.0 m/s and the particle diameter 0.45 mm. From Figure 6, it can be found that the rebound and ejected liftoff angular velocities respectively obey different functions. The former looks like Gumbel distribution with a cusp in the peak, and the latter a normal one. The liftoff angular velocities are usually lower than 500 rev/s. The lower the angular velocities are, the larger the probability density. In addition, the probability densities of the rebound liftoff angular velocity ranged respectively in (122.4, 497.7) and (119.3, 542.2) are smaller than the ejected ones. In the other regions, the situation is reversed. It states that the rebound particles are easier to possess higher or smaller liftoff angular velocity than the ejected ones to a certain extent. It should be noted that the rebound and ejected particles are able rotating not only in the clockwise but also in the anticlockwise. Their probabilities shown in Figure 6 are actually not equal. For the rebound particle, the probability of clockwise rotation is a little larger than the one of anticlockwise rotation.

[20] For a given wind speed and a given particle diameter, the probability densities of the rebound and ejected liftoff angular velocities are displayed in Figures 7 and 8. It can be found, from Figures 7 and 8, that the wind speed and the particle diameter have an influence on the probability density. In the range of the lower liftoff angular velocity, the probabilities of the angular velocities for the particles with 0.45 mm in diameter are larger than the ones with 0.25 mm and 0.35 mm in diameter, respectively, (see Figure 7). However, this kind of situation reverses with the increasing the liftoff angular velocity. In other words, the probabilities of the angular velocities for the particles with 0.45 mm in diameter are smaller than the ones with 0.25 mm and 0.35 mm in diameter, respectively, in the range of the higher liftoff angular velocity. Moreover, it can be found, from Figure 8, that the probabilities of the lower liftoff angular velocity reduce and the probabilities...
of the larger one increase with the increasing wind speed for particles with 0.45 mm in diameter.

[21] In order to illuminate the effect of wind speed and particle diameter on the lift off angular velocity in detail, the probability densities of the lift off angular velocities respectively taken as 100 rev/s, 400 rev/s and 600 rev/s versus the wind speed and the particle diameter are respectively plotted in Figures 9 and 10. From Figures 9a and 9b, it can be found that the probability densities of both the rebound and the ejected lift off angular velocities reduce with the increasing angular velocities. For larger lift off angular velocities, the ejected probability densities are usually greater than the rebound ones for particles with 0.35 mm in diameter, but it is not always sure for particles with 0.45 mm in diameter. When the lift off angular velocity of particle is 400 rev/s, the probability density values of the rebound and the ejected velocity vary alternately when wind speeds are smaller than 12 m/s. Moreover, the effect of the wind speed on the probability densities of the lift off angular velocities is related to the particle diameter. For particles with 0.35 mm in diameter, the significant effect happens when the wind speeds are about 10.0–14.0 m/s, which can be called the sensitive wind speeds. With the increasing diameter, the sensitive wind speeds are about 8.0–12.0 m/s and smaller than ones for smaller particle diameter. From Figures 10a and 10b, it can be found that the probability density values of rebound lift off angular velocity are usually greater than the ejected ones for smaller angular velocities and smaller particles. However, the effect of the particle diameters on the probability densities of the lift off angular velocity becomes not only significant but also complicate after the diameter is larger than 0.3 mm.

6. Conclusions

[22] The analytical formulas of the rebound and ejected lift off angular velocities of particles in a windblown sand flux are derived based on a stochastic particle-bed collision model presented in this paper. In this collision model, the incident velocity and angle of an incident particle, the impact angle between the incident and the impacted particles in the sand bed and its creep velocity as well as the contact angle between the impacted particle and other particles in the bed are taken as random variables. Having discussed the probability density functions of these random variables both in analysis and in numeric, the probability density functions of the lift off rebound and ejected angular velocities are calculated and are confirmed to agree well with the experimental ones measured both in this paper and in previous researches. The theoretical results show that the lift off rebound angular velocity obeys a distribution with a cusp in the peak and the ejected one distributes with the normal density function. The probability profiles of the clockwise and anticlockwise rotation for both the rebound and ejected particles are asymmetrical. The effect of the particle diameter on the probability density becomes more significant with the increasing the diameter. The probability density becomes sensitive only about a part of wind speeds. The smaller the particles, the larger the sensitive wind speeds are. The probability density of a given angular velocity almost does not change when the wind speed is out of its sensitive region.

Appendix A: Equations and Conditions for the Collision Model

[23] A collision can be usually divided into two processes, compression and recovery. After the rotation momentum and tangential impulse are taken into account and the theorems of impulsive momentum and impulsive moment are applied to each process, 12 algebraic equations are obtained as follows: Equations for compression process:

\[
S_{1r}' = m \left( U_{At} \sin \beta + U_{Ay} \cos \beta - V_{r} \sin (\beta - \theta) \right)
\]

\[
S_{1a}' = m \left( U_{At} \sin \beta - U_{At} \cos \beta + V_{a} \cos (\beta - \theta) \right)
\]

\[
S_{1r}' R = m \omega R^2 / 2
\]

\[
S_{2r} \cos \alpha - S_{2a} \sin \alpha + S_{1a} \cos \beta - S_{1r} \sin \beta = m (U_{Bx} - V_{B} \cos \alpha)
\]

\[
S_{2a} \cos \alpha - S_{2a} \sin \alpha - S_{1a} \sin \beta - S_{1r} \cos \beta = m (U_{Bx} - V_{B} \sin \alpha)
\]

\[
(S_{2r} + S_{1r}) R = m \omega R^2 / 2
\]

Equations for whole collision process:

\[
S_{1r} = m \left( U_{At} \sin \beta + U_{Ay} \cos \beta - V_{r} \sin (\beta - \theta) \right)
\]

\[
S_{1a} = m \left( U_{At} \sin \beta - U_{At} \cos \beta + V_{a} \cos (\beta - \theta) \right)
\]

\[
S_{1r} R = m \omega R^2 / 2
\]

\[
S_{2r} \cos \alpha - S_{2a} \sin \alpha + S_{1a} \cos \beta - S_{1r} \sin \beta = m (U_{Bx} - V_{B} \cos \alpha)
\]

\[
S_{2a} \cos \alpha - S_{2a} \sin \alpha - S_{1a} \sin \beta - S_{1r} \cos \beta = m (U_{Bx} - V_{B} \sin \alpha)
\]

\[
(S_{2r} + S_{1r}) R = m \omega R^2 / 2
\]

Here, \( m \) and \( R \) are the effective mass and radius of a sand grain, respectively; \( U \) and \( \omega \) are the linear velocity and the angular velocity of particles at the end of each process, \( S \) is the impulse momentum of interact forces at contact points; the superscript prime on a quantity implies that the quantity belongs to the compression process; the subscripts \( A \) and \( B \)
as well as x and y are respectively adopted to identify the particles and the components of quantities in the Cartesian coordinates, the subscripts τ and n represent the tangential and normal components of the impulses, respectively.

[24] The coupling conditions at the points J₁ and J₂ are listed as follows:

\[ U'_{x} \cos \beta - U'_{y} \sin \beta = U'_{b_{x}} \cos \beta - U'_{b_{y}} \sin \beta \]
\[ \omega'_{b} R + U'_{b_{x}} \cos \alpha + U'_{b_{y}} \sin \alpha = 0 \]
\[ \omega'_{A} R + U'_{A_{x}} \cos \beta + U'_{A_{y}} \sin \beta = -\omega'_{b} R + U'_{B_{x}} \cos \beta + U'_{B_{y}} \cos \beta \]
\[ U'_{b_{x}} \cos \alpha - U'_{b_{y}} \sin \alpha = 0 \]

(A13)

[25] From the collision theory, the normal and tangential restitution coefficients, denoted k₁ and k₂, respectively, can be given as

\[ k_{1} = \frac{U'_{b_{x}} \sin \beta - U'_{A_{x}} \cos \beta - \left( U'_{A_{x}} \sin \beta - U'_{A_{x}} \cos \beta \right)}{U'_{A_{x}} \sin \beta - U'_{A_{x}} \cos \beta + V_{A} \cos (\beta - \theta_{0})} \]
\[ k_{2} = \frac{U'_{A_{x}} \cos \beta + U'_{A_{y}} \sin \beta + \omega_{A} R_{1} - \left( U'_{b_{x}} \cos \beta + U'_{A_{x}} \sin \beta + \omega_{b} R_{1} \right)}{U'_{b_{x}} \cos \beta + U'_{A_{x}} \sin \beta + \omega_{b} R_{1} - V_{0} \sin (\beta - \theta_{0})} \]

(A14)

Here, restitution coefficients k₁ and k₂ are usually taken by experiments. For simplicity, the assumption of k₁ = k₂ = k is adopted in this paper.

[26] The additional equations are listed as follows:

\[ S_{1n} = (1 + k_{1}) S'_{n_{A_{x}}} \]
\[ S_{2n} = (1 + k_{1}) S'_{n_{A_{y}}} \]
\[ S_{1r} = (1 + k_{2}) S'_{r_{A_{x}}} \]
\[ S_{2r} = (1 + k_{2}) S'_{r_{A_{y}}} \]

(A15)

[27] Having solved the equations (A1)–(A12) with the conditions (A13)–(A15), the analytical expressions of lift-off angular velocity of sand particles can be obtained.

Notation

A Incident sand particle.
Rₐ Effective radius of incident sand particle.
B Sand particle impacted.
R₉ Effective radius of sand particle impacted.
J₁ Collision point.
θ₀ Incident angle.
β Impact angle.
C Particle bearing the resultant action of sand particles in sand bed.
J₂ Action point of sand particle and other sand particle in sand bed.
α Contact angle.
k Coefficient of restitution.
ωₐ Angular velocity of sand particle A.
ωₙ Angular velocity of sand particle B.
fₐ Angular density function of angular velocity of sand particle A.

fₐ Probability density function of angular velocity of sand particle B.
\[ f_{x_{A}}(x_{A}, y_{A}, \theta_{A}, \phi_{A}) \] Joint probability density function of \( V_{B_{x}}, V_{B_{y}}, \theta_{B}, \phi_{B} \).
\[ J_{x_{C}} \] Jacobian determinant of \( \omega_{A} \) with respect to \( V_{B_{x}} \).
\[ J_{x_{C}} \] Jacobian determinant of \( \omega_{B} \) with respect to \( V_{B_{x}} \).
\[ f_{x_{C}} \] Probability density function of \( V_{B_{x}} \).
\[ f_{x_{C}} \] Probability density function of \( V_{B_{y}} \).
\[ f_{x_{C}}(\beta, \theta_{0}) \] Joint probability density function of \( \beta \) and \( \theta_{0} \).
\[ f_{x_{C}}(\beta, \theta_{0}) \] Conditional probability density function of \( \beta \) under \( \theta_{0} \).
\[ f_{x_{C}}(\omega_{A}, \omega_{B}) \] Probability density function of \( \omega_{A} \).
\[ \omega_{A} \] Liftoff angular velocity.
\[ \omega_{B} \] Liftoff angular velocity.

(A28)

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