EDG ADVANCING RULES FOR INTERSECTING
SPHERICAL CONVEX POLYGONS*

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ABSTRACT
In this paper, we propose new rules of advancing edges for computing the intersection of
a pair of convex polygons in the plane. These rules have no ambiguities when extended
into the spherical surface, differently from those of O’Rourke et al.4 Finally, we design
a linear-time algorithm for computing the intersection of a pair of spherical convex
polygons, and prove its correctness.

Keywords: Computational geometry; intersection; spherical algorithm.

1. Introduction
The problem of finding the intersection of spherical convex polygons has often
appeared in CAD/CAM applications such as NC machining and mould design, as
pointed by Chen et al.1,2,3 For example, in NC machining, the problem of finding
the orientation of a workpiece for a 3-axis tool to machine the maximum number
of pocket surfaces in one set-up can be reduced to that of finding the maximal in­
tersection of spherical convex polygons that represent the visibility maps of those
surfaces. In mould design, a direction along which two mould pieces can be sepa­
rated safely is obtained by finding the intersection of all spherical convex polygons,
each of which represents the visibility map of a pocket surface. This paper presents
an efficient and practical algorithm for computing the intersection between a pair
of spherical convex polygons. This algorithm can be used as a primitive for solving
more general problems as stated above.

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In the planar case, several algorithms\textsuperscript{4,6,7,8} have been proposed for computing the intersection of a pair of convex polygons. For adapting those planar algorithms for the spherical case, one may project a pair of spherical convex polygons onto two parallel planes through central projection\textsuperscript{5} to obtain their planar images that are convex polygons. Then, we would apply the planar algorithms of Refs. [6,7,8] to obtain their intersection and project back to the original space to find the solution. From theoretical point of view, this projection technique can improve the efficiency of spherical algorithms including the intersection of a pair of spherical convex polygons. From the practical view of implementation, the projection causes computation burden such as a numerical problem of unbounded values. Therefore, we avoid this approach to adapt the planar algorithms for the spherical space, instead.

The algorithm of Shamos and Hoy\textsuperscript{6,7} subdivides the plane into a sequence of infinite strips using the set of parallel lines passing through the vertices of the polygons, so that the portion of each polygon in every strip is simple enough for computing their intersection. We can easily transform the algorithm into its spherical counterpart. That is, an arbitrary point on the spherical space is chosen to emit a set of great circles from it to all vertices, which gives rise to a sequence of spherical strips.

The algorithm of Toussaint\textsuperscript{8} combines two well-known geometric algorithms: one for finding the convex hull of a polygon and the other for triangulating a polygon. At a first glance, this algorithm would be transformed to its spherical version easily. However, it is not obvious how we can achieve it unless the polygons are contained in a hemisphere.

Finally, we focus our attention on the algorithm of O’Rourke et al.,\textsuperscript{4} which is known as the simplest to implement. To compute the intersection of two convex polygons, the algorithm examines a pair of edges from different polygons at each step to decide which edge to choose for moving to its next one along the boundary of the polygon containing it. This edge advancing depends on the geometric relationship between the pair of edges under examination. However, this geometric relationship in the planar space is different to that in the spherical space. We improve the edge advancing mechanism to adapt this algorithm for the spherical convex polygon intersection problem.

Apart from the numerical problem of the projection technique, note that the edge advancing mechanism of the planar algorithm\textsuperscript{4} can be applied into the projected space, only if two spherical polygons are projected onto a single plane, that is, they are contained in a hemisphere. This is another strong motivation for developing a new edge advancing mechanism.

2. Previous Advancing Mechanism

2.1. Chasing Rules of O’Rourke for Advancing Edges in the Plane

Let $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$ be two convex polygons whose vertices are specified in the counter-clockwise order. An edge $\overrightarrow{p_{i-1}p_i}$ is shortly written by $\overrightarrow{p_i}$. $LHS(\overrightarrow{p_i})$ and $RHS(\overrightarrow{p_i})$, respectively, describe the left and right open
half-spaces that are divided by the directed line determined by the edge $\overline{p_i}$.

For more definitions and terminology, we follow the ones inspired by Ref. [5]. Let $P^*$ be the polygon $P \cap Q$, that is, the intersection of $P$ and $Q$, as illustrated in Figure 1. We denote $\{a_1, \ldots, a_k\}$ the intersection points of edges of $P$ and $Q$ in the counter-clockwise order. A chain is defined as a sequence of vertices of a polygon. If we traverse the vertices of $P^*$, the chains of $P$ alternate with those of $Q$ at each of $\{a_1, \ldots, a_k\}$. If a chain of $P$ between $a_{i-1}$ and $a_i$ lies on the boundary of $P^*$, then a chain of $Q$ between $a_{i-1}$ and $a_i$ wraps around it in the exterior of $P$ and vice versa. The polygon formed by the two intersection points and the two chains between them is called a sickle. The chain lying on the boundary of $P^*$ is said to be internal of the sickle while the other is external. The two points $a_{i-1}$ and $a_i$, respectively, are called initial and terminal of the sickle. Note that $a_i$ is also the initial of another sickle formed between $a_i$ and $a_{i+1}$. We say that an edge $\overline{p_i}$ belongs to a sickle (chain) if and only if either or both of the vertices $p_{i-1}$ and $p_i$ belong to the sickle (chain). Hence, an edge containing an intersection point belongs to two sickles (chains).

We define the chasing relation between edges, as illustrated in Figure 2(a). We assume that $P$ and $Q$ lie on the $xy$ plane in the right-handed 3D coordinate system. In expressions, the symbols $\neg$, $\wedge$ and $\lor$ are the boolean operations of negation, intersection, and union, respectively.
Definition 1  
\[ \text{Chase}(\vec{q}_j, \vec{p}_i) = (Z(\vec{p}_i \times \vec{q}_j) > 0 \land q_j \in \text{RHS}(\vec{p}_i)) \lor (Z(\vec{p}_i \times \vec{q}_j) < 0 \land q_j \in \text{LHS}(\vec{p}_i)) \],  
where \(Z(\vec{p}_i \times \vec{q}_j)\) means the z-component of the cross product, \(\vec{p}_i \times \vec{q}_j\).

O'Rourke et al.\(^4\) have developed edge-advancing rules by using the chasing relation between edges and their relative positions, as shown in Table 1, which will be called the chasing rules. For proving that all intersection points are correctly found by advancing edges one by one under the chasing rules, they have shown that at least one intersection point can be found with these rules, and then given an intersection point, its next one along the boundary of \(P^*\) is guaranteed to be found.

**Table 1.** Chasing rules Ref. [4].

<table>
<thead>
<tr>
<th>(\text{Chase}(\vec{q}_j, \vec{p}_i))</th>
<th>(-\text{Chase}(\vec{p}_i, \vec{q}_j))</th>
</tr>
</thead>
</table>
| **Chase(\(\vec{q}_j, \vec{p}_i\))** | **Advances \(\vec{p}_i\), if \(p_i \in \text{RHS}(\vec{q}_j)\).**  
**Advances \(\vec{q}_j\), otherwise.** |
| **\(-\text{Chase}(\vec{p}_i, \vec{q}_j)\)** | **Advances \(\vec{q}_j\).**  
**Advances \(\vec{p}_i\), if \(p_i \in \text{RHS}(\vec{q}_j)\).**  
**Advances \(\vec{q}_j\), otherwise.** |

2.2. **Ambiguities of Chasing Rules in the Spherical Space**

For considering the extension of the edge advancing mechanism into the spherical space, we begin by describing some terminologies in the spherical space. The space on the boundary of the unit sphere centered at origin is described as \(S^2 = \{p : ||p|| = 1\}\). A great circle on \(S^2\) is determined by the intersection of the unit sphere with a plane containing the origin. An arc joining two distinct points \(p\) and \(q\) on \(S^2\) is the shorter of the two arcs in the great circle containing \(p\) and \(q\). A spherical polygon is a sequence of arcs joining points on \(S^2\), whose vertices are specified in the counter-clockwise order. A spherical polygon \(P\) is said to be convex, if, for any two points in the interior of \(P\), the arc joining the two points lies entirely in the interior of \(P\). The central projection of a point \(p\) on \(S^2\) into a plane is the mapping of \(p\) into the intersection point between the plane and the ray from the origin to \(p\). This projection establishes the correspondence between an arc (great circle) on \(S^2\) and an edge (line) in the plane.

![Fig. 3. Ambiguities on the sphere.](image-url)
The chasing rules are very elegant in the plane, but have two ambiguities when they are extended into $S^2$. First, a pair of arcs on $S^2$ always chase each other in general. However, there are three kinds of chasing relation between a pair of edges in the plane; they chase each other, only one edge chases the other, or neither of them chases each other. Note that the intersection of two distinct great circles gives two points. If we project the spherical convex polygons into a plane, the chasing relation between a pair of the projected arcs depends on the choice of projection planes, as illustrated in Figure 3(a) and (b). The arc $q_j$ chases the arc $p_i$ in Figure 3(a), while $p_i$ chases $q_j$ in Figure 3(b).

Second, the chasing relation between arcs and their relative position can be strictly symmetric, that is, two arcs chasing each other can be simultaneously on the left or right side of each other, as illustrated in Figure 3(c) and (d). This situation never occurs in the plane; if two edges in the plane chase each other, one is on the left of the other while the other on the right of the one. This is true when neither of them chases each other.

3. New Advancing Mechanism

3.1. Facing Rules for Advancing Edges in the Plane

Motivated from the observation of Section 2.2, we propose new edge advancing rules in the plane, and extend them into the spherical space in Section 3.2. The main idea of the new rules is based on the facing relation between edges and their relative positions, as illustrated in Figure 2(b). The facing relation is mathematically formulated as follows.

**Definition 2**

$$\text{Face}(q_j, p_i) = (p_{i-1} \in \text{LHS}(q_j)) \land (p_i \in \text{RHS}(q_j)) \land (q_j \in \text{RHS}(p_i)) \lor (p_{i-1} \in \text{RHS}(q_j)) \land (p_i \in \text{LHS}(q_j)) \land (q_j \in \text{LHS}(p_i))$$

The newly developed rules by using the facing relation are shown in Table 2, which will be called the facing rules.

<table>
<thead>
<tr>
<th>Face$(q_j, p_i)$</th>
<th>$\neg$Face$(p_i, q_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule 1: ___</td>
<td>Rule 2: Advance $q_j$.</td>
</tr>
<tr>
<td>Rule 3: Advance $p_i$.</td>
<td>Rule 4:</td>
</tr>
<tr>
<td>(1) Advance $p_i$, if $p_i \in \text{RHS}(q_j)$.</td>
<td></td>
</tr>
<tr>
<td>(2) Advance $q_j$, if $q_j \in \text{RHS}(p_i)$.</td>
<td></td>
</tr>
<tr>
<td>(3) Advance $p_i$, otherwise.</td>
<td></td>
</tr>
</tbody>
</table>

We do not mention the degenerate cases such that the endpoint of an edge is on other edge, since they can be treated similarly with Ref. [4]. We begin by the following definition for establishing the correctness of the facing rules.

**Definition 3**

$P$ and $Q$ are said to be geared with respect to $p_i$ and $q_j$, if $(q_j \in \text{RHS}(p_i)) \lor \text{Face}(p_i, q_j)$ ($p_i$ and $q_j$ are interchangable.), where $p_i \in P$ and $q_j \in Q$ are current edges during the advancing.
The geared state does not mean that $\overline{p_i}$ and $\overline{q_j}$ belong to one sickle, but, clearly, $P$ and $Q$ are geared if current edges intersect each other. Hence, we prove Lemma 1, instead of showing that if one intersection point is found, the next one is guaranteed to be found also.

**Lemma 1** If $P$ and $Q$ that intersect each other are geared, the first intersection point from current edges $\overline{p_i} \in P$ and $\overline{q_j} \in Q$ will be properly found.

**Proof.** Without loss of generality, assume that $q_j \in RHS(\overline{p_i})$, since other geared states can be proved symmetrically. We denote $\overline{p_k} \in P$ and $\overline{q_l} \in Q$ the edges containing the first intersection point $w$ from current edges.

At the first time, the edges of $Q$ advance by Rule 4-2 of Table 2 until some $q'$ such that $\text{Face}(\overline{p_i}, q')$ holds true is found. And then, the edges of $P$ advance by Rule 3 until some $q' \in RHS(\overline{p_i})$ is found. These two sequences of advancings are alternately repeated until some edge $q''$ such that $\text{Face}(\overline{p_k-1}, q'')$ holds true is found. If $q'' = q_l$, as illustrated in Figure 4(a), $\overline{p_k-1}$ advances by Rule 3 and the intersection point $w$ is found. Otherwise, as illustrated in Figure 4(b), $\overline{p_k-1}$ advances, and then, the edges of $Q$ advance again by Rule 4-2 until the intersection point $w$ is found. □

In order to show that at least an intersection point always found, we show that ungeared states always change into geared states by the facing rules.

**Lemma 2** If $P$ and $Q$ with current edges $\overline{p_i}$ and $\overline{q_j}$ are ungeared, they will become geared in the sickle containing $\overline{q_j}$.

**Proof.** The ungeared state is equivalent to the condition of $\neg \text{Face}(\overline{q_j}, \overline{p_i}) \land \neg \text{Face}(\overline{p_i}, \overline{q_j}) \land (p_i \in LHS(\overline{q_j})) \land (q_j \in LHS(\overline{p_i}))$. Hence, only Rule 4-3 of Table 2 is applied into the advancing until the state is broken, i.e., a geared state is reached. The sickle in the geared state contains the edge $\overline{q_j}$, since $\overline{q_j}$ remains stationary while the edges of $P$ advance until the geared state is reached. □

We compare the chasing rules with the facing rules. Once a geared state is reached, from that time, the order of edge advancing under the chasing rules is same with that under the facing rules. But, until a geared state is reached, the facing rules advance a fixed one between $P$ and $Q$ while the chasing rules advance
a chasing one between them. Hence, the advancing before being geared under the facing rules iterates two times on average than that under the chasing rules. However, the establishment of the correctness of the new rules is simpler, and their ultimate goal is to be able to run on the spherical space.

3.2. Extension of Facing Rules into the Spherical Space

In the spherical space, $LHS(\overline{p_i})$ and $RHS(\overline{p_i})$, respectively, denotes the left and right hemispheres that are divided by the great circle determined by an arc $\overline{p_i}$. Then, the facing relation between arcs is samely defined with Definition 2. As illustrated in Figure 5, (a) and (c) are facing, but (b) and (d) are not facing. Note that the facing relation between arcs does not change even though they are projected into any plane.

The facing relation on $S^2$ has a special case differently to that in the plane; two arcs can face each other like two snakes biting the tail of each other, as illustrated in Figure 3(c) and (d). We call this relation that the two arcs are *meeting*. For treating the meeting case, we extend the facing rules by replacing Rule 1 of Table 2 with the following.

**Rule 1:**

1. Advance either of $\overline{p_i}$ and $\overline{q_j}$, which has previously arrived, if geared.
2. Advance $\overline{p_i}$, if ungeared.

We call Rule 1-1 the *symmetric rule*, since it does not advance a fixed one between two arcs, but advances either of them symmetrically.

**Lemma 3** On the spherical space, if $P$ and $Q$ that intersect each other are geared, the first intersection point from current edges $\overline{p_i} \in P$ and $\overline{q_j} \in Q$ will be properly found.

**Proof.** In order to establish the correctness of the extended facing rules, it is sufficient to concentrate on Rule 1, since the proof of other rules is same with that of Lemma 1.

Assume that the first meeting has occurred at two arcs $\overline{p_i}$ and $\overline{q_j}$. There are two cases of the meeting: $(p_i \in LHS(\overline{q_j})) \land (q_j \in LHS(\overline{p_i}))$ (Figure 3(c)) and $(p_i \in RHS(\overline{q_j})) \land (q_j \in RHS(\overline{p_i}))$ (Figure 3(d)). In the latter case, either of $\text{Face}(\overline{p_{i-1}}, \overline{q_j})$ or $\text{Face}(\overline{q_{j-1}}, \overline{p_i})$ should be true, since $(p_{i-1} \in LHS(\overline{q_j})) \land (q_{j-1} \in LHS(\overline{p_i}))$. If $\text{Face}(\overline{p_{i-1}}, \overline{q_j})$ is true, the arcs of $Q$ will advance until an intersection point with $p_i$ is found, or vice versa. This means that $\overline{p_i}$ and $\overline{q_j}$ belong to one sickle.
In the former case, $p_i$ and $q_j$ may not belong to one sickle, since $p_i \in LHS(q_{j-1})$ or $q_j \in LHS(p_{i-1})$ may be true. However, any meeting never occurs until the first intersection point is found, since all arcs of $P$ and $Q$ to that time lie in the area of $LHS(p_i) \cap LHS(q_j)$, which is hemispherical. Note that two meeting arcs are not contained in any hemisphere.

Furthermore, once an intersection point is found, current edges during advancing from that time will always belong to one sickle. Therefore, it is sufficient to prove that the geared state will continue by the symmetric rule when current meeting arcs $p_i$ and $q_j$ belong to one sickle $L$.

Without loss of generality, suppose that the external of $L$ is a part of $P$, and its internal is a part of $Q$. Let $v$ and $w$, respectively, be the initial and the terminal of $L$. We denote $p^v$ and $q^v$ the arcs containing $v$, and $p^w$ and $q^w$ the arcs containing $w$. We show that there can be only two cases of $(q_j - q^w) \cap (q_j = q^w)$, and the geared state continues if the symmetric rule is applied into current arcs $p_i$ and $q_j$.

If $q_j \in LHS(p_i)$ (Figure 6(a)), there exists an arc $p^v$ before $p_i$, which contains $v$ with $q_j$, since $p_i$ and $q_j$ belong to one sickle and all arcs of $P$ lie in $LHS(p_i)$. That is, $q_j = q^v$. Similarly, we can know that there exists an arc $p^w$ after $p_i$, which contains $w$ with $q_j$, if $q_j \in RHS(p_i)$ (Figure 6(b)). That is, $q_j = q^w$.

In the case of $q_j = q^v$, we can observe that $p_i$ has recently arrived since $p_{i-1} \in RHS(q_j)$. Hence, the geared state continues after advancing $q_j$ by Rule 1 of Table 3. In the case of $q_j = q^w$, assume that $p_i$ has recently arrived. This assumption implies $(p_{i-1} \in RHS(q_j)) \cup (p_{i-1} \in LHS(q_j)) \cap (q_{j-1} \in LHS(p_i))$. This is the contradiction to that the initial of $L$ has been found, i.e., that $P$ and $Q$ had been geared. At this time, $p_i$ advances and the geared state continues.

This proof implicitly implies that a geared state may be broken if the symmetric rule is not applied into meeting arcs.

**Lemma 4** On the spherical space, if $P$ and $Q$ with current edges $p_i$ and $q_j$ are ungeared, they will become geared in the sickle containing $q_j$.

**Proof.** The proof is straightforward, since Rule 4-3 that is applied into the meeting case is same with Rule 1-2.
Before finishing our proof, we show why the symmetric rule should not be applied into the meeting arcs until a geared state is reached. If the symmetric rule is applied to them before a geared state is reached, the arcs of $P$ and $Q$ may advance infinitely. For an example, assume that given a pair of $P = \{p_1, \ldots, p_{2n}\}$ and $Q = \{q_1, \ldots, q_n\}$, $p_{2i}$ meets $q_i$, and $p_{2i-1} \in RHS(q_i) \cap q_{2i} \in RHS(p_{2n})$, for all $i = 1, \ldots, n$. If we apply the symmetric rule into this case, the advancing sequence $\{(q_i, p_i, p_{2i}), (q_i, p_{i+1}, p_{2i+1}) \}$ may be infinitely repeated. It is really possible to construct $P$ and $Q$ with such geometric relationships between edges.

Based on the extended facing rules, we describe a linear-time algorithm for intersecting a pair of spherical convex polygons as follows.

**Procedure INTERSECTION** ($\{p_1, \ldots, p_m\}, \{q_1, \ldots, q_n\}$)

**Input**: A pair of spherical convex polygons $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$

**Output**: A polygon of $P \cap Q$

Choose $p_i$ and $q_j$ arbitrarily.

while $(p_i \in LHS(q_j)) \wedge (q_j \in LHS(p_i)) \vee \text{Face}(p_i, q_j) \wedge \text{Face}(q_j, p_i))$

Endwhile

do

if $p_i$ intersects with $q_j$ then

if the first intersection then Initialize advancing counters of $P$ and $Q$ into 0.

Output the intersection point of $p_i$ and $q_j$.

endif

if $\text{Face}(p_i, q_j) \wedge \text{Face}(q_j, p_i)$ then

Advance either of $p_i$ or $q_j$, which has previously arrived.

else if $\text{Face}(q_j, p_i)$ then

Advance $q_j$.

else if $\text{Face}(p_i, q_j)$ then

Advance $p_i$.

else

if $q_j \in RHS(p_i)$ then

Advance $q_j$.

else

Advance $p_i$.

endif

until ((counter of $P = m$) \wedge (counter of $Q = n$) \vee (counter of $P = 2m$) \vee (counter of $Q = 2n$).

if $P \subset Q$ then Output $P$.

else if $Q \subset P$ then Output $Q$.

endProcedure INTERSECTION.

Finally, we obtain the following result.

**Theorem 1** Using the extended facing rules of edge advancing, the intersection of a pair of spherical convex polygons $P$ and $Q$ can be computed in $O(m + n)$ time, where $m$ and $n$ are the numbers of arcs of $P$ and $Q$, respectively.

**Proof.** In the procedure INTERSECTION, the while loop for reaching a geared state is repeated at most $m$ times. In the do loop after reaching the geared state, the first intersection point is found in the constant number of repeats if $P$ and $Q$ intersect each other, and then the repetition continues exactly $m + n$ times from that time. If $P$ and $Q$ do not intersect each other, the loop terminates before
being repeated $2m + 2n$ times. Hence, the intersection of $P$ and $Q$ is computed in $O(m + n)$ time.

4. Conclusion

In this paper, we proposed new edge-advancing rules for intersecting a pair of convex polygons in the planar space. The new rules are based on the facing relation between two edges and their relative position which can be extended into the spherical space. Finally, we established a practical algorithm that is based on the extended facing rules for intersecting a pair of spherical convex polygons in linear time.

References